Lie group methods

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Integrating differential equations on a manifold

Example: rotation kinematics in terms of a rotation matrix:

$$\dot{R} = R\hat{\omega}, \quad R \in SO(3), \quad \hat{\omega}x = \omega \times x$$

or in terms of a unit quaternion:

$$\dot{q} = \frac{1}{2} q \circ \begin{pmatrix} 0 \\ \omega \end{pmatrix}, \quad q \in \text{unit quaternions} \cong SU(2)$$

with ω the angular velocity in body frame.
Naive Euler step:

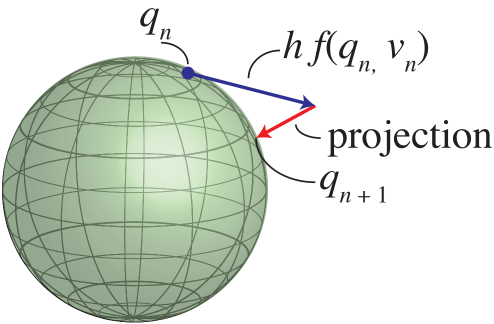
$$R_{n+1} = R_n + R_n \hat{\omega}_n h \notin SO(3)$$

In general:

$$\dot{q} = f(q, v) \text{ with } q \text{ on a manifold}$$

A quick hack: projection

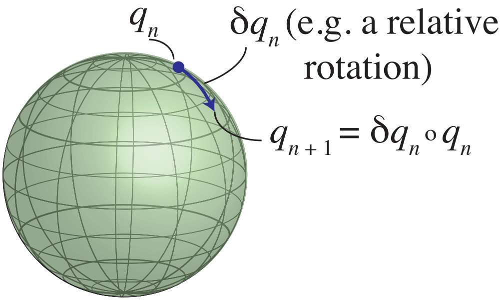
$$q_{n+1} = P\left(q_n + hf\left(q_n, v_n\right)\right)$$



- Easiest to implement
- Accuracy?
- Not the natural thing to do

Using the group multiplication operation

Better: represent the integration step as a group member and compose.



Example:

- Initial orientation: $R_n \in SO(3)$
- Step: $\delta R_n \in SO(3)$
- Final orientation $R_{n+1} = \delta R_n R_n \in SO(3)$

Recap: explicit Runge-Kutta methods

Dynamics:

$$\dot{x} = f(t, x)$$

Standard explicit Runge-Kutta method with s stages:
Big step:

$$x_{n+1} = x_n + h \sum_{i=1}^{s} b_i k_i$$

Computed from s baby steps:

$$k_i = f(t_i, y_i)$$

$$t_i = t_n + c_i h$$

$$y_i = x_n + h \sum_{j=1}^{i-1} a_{ij} k_i$$

Recap: explicit Runge-Kutta methods

Example: Heun's method (improved Euler):

$$\begin{array}{c|cc} c_1 & & \\ c_2 & a_{21} & \\ \hline & b_1 & b_2 \end{array} = \begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array} \quad \text{(Butcher tableau)}$$

$$\begin{aligned} k_1 &= f\left(t_n, x_n\right) \\ k_2 &= f\left(t_n + 1h, x_n + 1hk_1\right) \\ x_{n+1} &= x_n + \frac{1}{2}hk_1 + \frac{1}{2}hk_2 \end{aligned}$$

Recap: explicit Runge-Kutta methods

Example: classical 4th order Runge-Kutta method:

$$\begin{array}{c|cccc} c_1 & & & & \\ c_2 & a_{21} & & & \\ c_3 & a_{31} & a_{32} & & \\ c_s & a_{31} & a_{32} & a_{33} & \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array} = \begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Four stages (evaluations of the dynamics)
Total accumulated error: $O\left(h^4\right)$.

Crouch-Grossman methods

Regular RK:	Crouch-Grossman:
$x_{n+1} = x_n + h\sum_{i=1}^{s} b_i k_i$	$x_{n+1} = \exp\left(hb_s k_s\right) \circ \cdots \circ \exp\left(hb_1 k_1\right) x_n$
$k_i = f\left(t_i, y_i\right)$	$k_i = f\left(t_i, y_i\right)$ (returns a tangent vector)
$t_i = t_n + c_i h$	$t_i = t_n + c_i h$
$y_i = x_n + h\sum_{j=1}^{i-1} a_{ij} k_i$	$y_i = \exp\left(ha_{i,i-1} k_{i-1}\right) \circ \cdots \circ \exp\left(ha_{i1} k_1\right) x_n$

Crouch-Grossman methods

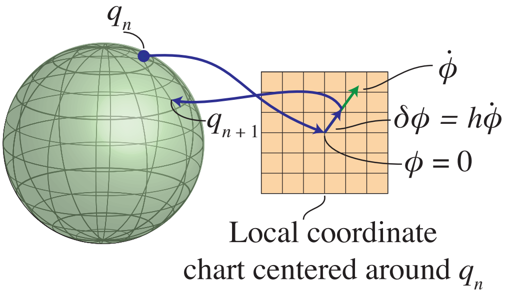
Coefficients for third-order method:

$$\begin{array}{c|ccc} 0 & & & \\ 3/4 & 3/24 & & \\ 17/24 & 119/216 & 17/108 & \\ \hline & 13/51 & -2/3 & 24/17 \end{array}$$

- Need to use very specific coefficients!
- Coefficients for higher order methods are hard to compute.
- Fourth order method requires five stages!

Munthe-Kaas methods

Instead of making steps on the manifold using group multiplication, rewrite dynamics in terms of local coordinates around previous state.



Munthe-Kaas methods

Example: rotation kinematics.

- Initial local coordinates around R_n: $\phi = 0$.
- Compute derivative in terms of local coordinates (Bortz equation):

$$\dot{\phi} = \omega + \frac{\phi \times \omega}{2} + \frac{1}{\|\phi\|^2}\left(1 - \frac{\|\phi\| \sin \|\phi\|}{2\left(1 - \cos \|\phi\|\right)}\right) \phi \times (\phi \times \omega)$$

- Compute stages in local coordinates
- Convert back to global coordinates: $R_{n+1} = \exp\left(\phi\right) R_n$

Munthe-Kaas methods

Properties:

- Can just use any integration method on local coordinate chart!
- Can use approximation of exponential coordinates.
- Performs better and is implemented more efficiently than Crouch-Grossman

References

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